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Nonlinear integral equations for thermodynamics of the $sl(r+1)$ Uimin–Sutherland model

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Abstract
We derive traditional thermodynamic Bethe ansatz (TBA) equations for the $sl(r+1)$ Uimin–Sutherland model from the T-system of the quantum transfer matrix. These TBA equations are identical to the those from the string hypothesis. Next we derive a new family of nonlinear integral equations (NLIEs). In particular, a subset of these NLIEs forms a system of NLIEs which contains only a *finite* number of unknown functions. For $r = 1$, this subset of NLIEs reduces to Takahashi's NLIE for the XXX spin chain. A relation between the traditional TBA equations and our new NLIEs is clarified. Based on our new NLIEs, we also calculate the high-temperature expansion of the free energy.

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1. Introduction

Thermodynamic Bethe ansatz (TBA) equations have been used to investigate thermodynamics of various types of solvable lattice model; see, for example, [1]. Traditionally, TBA equations have been derived by the string hypothesis [2, 3]. Several years ago, TBA equations for the supersymmetric t–J model and the supersymmetric extended Hubbard model were derived [4] from the T-system (a system of functional relations among transfer matrices) for the quantum transfer matrix (QTM) [5–9], which is independent of the string hypothesis. Later, TBA equations for the XXZ model in the regime $|\Delta| < 1$ [10], the $osp(1|2)$ model [11], and the $osp(1|2s)$ model [12] were derived by similar procedures in [4]. In addition, the TBA equation in [10] was analytically continued [13] to that for $|\Delta| \geqslant 1$. These TBA equations contain an infinite (or finitely many) number of unknown functions, and thus are not always easily treated. It is significant to simplify the TBA equations to tractable integral equations which contain only a finite number of unknown functions.

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In the context of the QTM method, for models related to algebras of rank one or two, nonlinear integral equations (NLIEs) with finite numbers of unknown functions were derived by Klümper and his collaborators [9, 14–18]. Although their NLIEs give the same free energy as the TBA equations, the derivations need trial and error for each model, which prevents their extension to higher rank case.

Another type of NLIE with only one unknown function was recently proposed for the XXZ spin chain by Takahashi [19]. We can also derive [13] Takahashi's NLIE from the T-system [20, 21] of the QTM. In view of this fact, we have derived [22] NLIEs with a finite number of unknown functions from our T-system [23] for the $osp(1|2s)$ model for arbitrary rank s. In this paper, we further derive NLIEs for the $sl(r+1)$ Uimin–Sutherland model [24, 25] with only a *finite* number (the number of rank r) of unknown functions from the T-system [21]. This is the first explicit derivation of this type of NLIE for a vertex model associated with $sl(r+1)$ for *arbitrary* rank r

In section 2, we introduce the $sl(r+1)$ Uimin–Sutherland model, and define the QTM and the T-system [21] related to this model. In section 3, we derive traditional TBA equations from the T-system defined in section 2 without using the string hypothesis. In section 4, we derive the NLIEs (4.6) and (4.7), which are our main results. The normalized fused QTMs $\left\{\mathcal{T}_m^{(a)}(v)\right\}$ ($a \in \{1, 2, \ldots, r\}$; $m \in \mathbb{Z}_{\geqslant 1}$: the fusion degree of the model) in the Trotter limit $N \to \infty$ defined in section 2 play the role of the unknown functions of these NLIEs. The traditional TBA equations (3.11), (3.13) and (3.16) are integral equations on the variables $\left\{Y_m^{(a)}(v)\right\}$, which connect with those $\left\{\mathcal{T}_m^{(a)}(v)\right\}$ for our new NLIEs (4.6) through the relation (3.1) (cf $\mathcal{T}_m^{(a)}(v) = \lim_{N\to\infty} \widetilde{T}_m^{(a)}(v)$). In this sense, we may view our new NLIEs (4.6) as TBA equations on the variables $\{\mathcal{T}_m^{(a)}(v)\}$. For the evaluation of the free energy, we need only the fundamental one $\mathcal{T}_1^{(1)}(v)$. Therefore, adopting an infinite number of fused QTMs $\left\{\mathcal{T}_m^{(a)}(v)\right\}$ as unknown functions seems to be superfluous. In fact, we see that these NLIEs (4.6) for $m = 1$ form a closed set of equations (4.7), which contains only a *finite* number of unknown functions $\left\{\mathcal{T}_1^{(a)}(v)\right\}_{1\leqslant a\leqslant r}$. For $r = 1$, this set of NLIEs (4.7) reduces to Takahashi's NLIE [19] for the XXX spin chain. On the other hand, NLIEs (4.6) for $m \geqslant 2$ have never been considered before, and thus they are new equations even in the case $r = 1$. Using our new NLIEs (4.7), we calculate the high-temperature expansion of the free energy in section 5. Section 6 is devoted to concluding remarks. Many calculations in this paper are parallel with those for the $osp(1|2s)$ case [12, 22], but we describe the calculations concisely, not so much for the reader's convenience, but because of the basic importance of the Uimin–Sutherland model.

2. *T*-system and QTM method

We introduce the $sl(r+1)$ Uimin–Sutherland model [24, 25], and we define the QTM [5–9] and the T-system [21] for this model. QTM analyses of the Uimin–Sutherland model can be found in [4, 18]. The classical counterpart of the $sl(r+1)$ Uimin–Sutherland model is a special case of the Perk–Schultz model, whose R-matrix [26] is given as

$$R_{\mu\nu}^{ab}(v) = v\delta_{ab}\delta_{\mu\nu} + \delta_{a\nu}\delta_{\mu b} \tag{2.1}$$

where $v \in \mathbb{C}$; $a, b, \nu, \mu \in \{1, 2, \ldots, r+1\}$. We define the QTM $t_1^{(1)}(v) = \left(\left(t_1^{(1)}\right)_{\{\alpha_1,\ldots,\alpha_N\}}^{\{\beta_1,\ldots,\beta_N\}}(v)\right)$ as

$$\left(t_1^{(1)}\right)_{\{\alpha_1,\ldots,\alpha_N\}}^{\{\beta_1,\ldots,\beta_N\}}(v) = \sum_{\{\nu_k\}} \exp\left(\frac{\mu_{\nu_1}}{T}\right) \prod_{k=1}^{\frac{N}{2}} R_{\alpha_{2k-1},\beta_{2k-1}}^{\nu_{2k-1},\nu_{2k}}(u+\mathrm{i}v)\widetilde{R}_{\alpha_{2k},\beta_{2k}}^{\nu_{2k},\nu_{2k+1}}(u-\mathrm{i}v). \tag{2.2}$$

Here $\widetilde{R}^{\mu,\nu}_{\alpha,\beta}(v) = R^{\beta,\alpha}_{\mu,\nu}(v)$; N is the Trotter number and is assumed to be even; $u = -\frac{J}{TN}$ (T is the temperature; J is a coupling constant, where $J > 0$ ($J < 0$) corresponds to the anti-ferromagnetic (ferromagnetic) regime); $\{\mu_a\}$ are chemical potentials; and the Boltzmann constant is set to 1. The free energy per site is expressed in terms of the largest eigenvalue $T^{(1)}_1(0)$ of the QTM (2.2) at $v = 0$:

$$f = -T \lim_{N\to\infty} \log T^{(1)}_1(0). \tag{2.3}$$

The eigenvalue formula $T^{(1)}_1(v)$ of the QTM (2.2) is imbedded into $T^{(a)}_m(v)$ for a fusion hierarchy of the QTM (cf Bazhanov–Reshetikhin formula in [29])

$$T^{(a)}_m(v) = \sum_{\{d_{j,k}\}} \prod_{j=1}^{a} \prod_{k=1}^{m} z\left(d_{j,k}; v - \frac{\mathrm{i}}{2}(a - m - 2j + 2k)\right) \tag{2.4}$$

where the summation is taken over $d_{j,k} \in \{1, 2, \dots, r+1\}$ such that $d_{j,k} \prec d_{j+1,k}$ and $d_{j,k} \preceq d_{j,k+1}$ $(1 \prec 2 \prec \cdots \prec r+1)$. The functions $\{z(a; v)\}$ are defined as

$$z(a; v) = \psi_a(v) \frac{Q_{a-1}\big(v - \frac{\mathrm{i}}{2}(a+1)\big) Q_a\big(v - \frac{\mathrm{i}}{2}(a-2)\big)}{Q_{a-1}\big(v - \frac{\mathrm{i}}{2}(a-1)\big) Q_a\big(v - \frac{\mathrm{i}}{2}a\big)} \quad \text{for} \quad a \in \{1, 2, \dots, r+1\} \tag{2.5}$$

where $Q_a(v) = \prod_{k=1}^{M_a} \big(v - v^{(a)}_k\big)$; $M_a \in \mathbb{Z}_{\geqslant 0}$; $Q_0(v) = Q_{r+1}(v) = 1$. The vacuum parts are given as follows

$$\psi_a(v) = \mathrm{e}^{\frac{\mu_a}{T}} \phi_+(v + \mathrm{i}\delta_{a,r+1}) \phi_-(v - \mathrm{i}\delta_{a,1}) \quad \text{for} \quad a \in \{1, 2, \dots, r+1\} \tag{2.6}$$

where $\phi_\pm(v) = (v \pm \mathrm{i}u)^{\frac{N}{2}}$. $\big\{v^{(a)}_k\big\}$ is a solution of the Bethe ansatz equation (BAE)

$$\frac{\psi_a\big(v^{(a)}_k + \frac{\mathrm{i}}{2}a\big)}{\psi_{a+1}\big(v^{(a)}_k + \frac{\mathrm{i}}{2}a\big)} = -\frac{Q_{a-1}\big(v^{(a)}_k + \frac{\mathrm{i}}{2}\big) Q_a\big(v^{(a)}_k - \mathrm{i}\big) Q_{a+1}\big(v^{(a)}_k + \frac{\mathrm{i}}{2}\big)}{Q_{a-1}\big(v^{(a)}_k - \frac{\mathrm{i}}{2}\big) Q_a\big(v^{(a)}_k + \mathrm{i}\big) Q_{a+1}\big(v^{(a)}_k - \frac{\mathrm{i}}{2}\big)}$$
$$\text{for} \quad k \in \{1, 2, \dots, M_a\} \quad \text{and} \quad a \in \{1, 2, \dots, r\}. \tag{2.7}$$

We can rewrite equation (2.7) as the Reshetikhin–Wiegmann BAE [27] in terms of the representation theoretical data:

$$-\prod_{j=1}^{N} \left(\frac{v^{(a)}_k - \frac{\mathrm{i}}{2} b^{(a)}_j - w^{(a)}_j}{v^{(a)}_k + \frac{\mathrm{i}}{2} b^{(a)}_j - w^{(a)}_j} \right) = \zeta_a \prod_{b=1}^{r} \frac{Q_b\big(v^{(a)}_k - \frac{\mathrm{i}}{2}(\alpha_a|\alpha_b)\big)}{Q_b\big(v^{(a)}_k + \frac{\mathrm{i}}{2}(\alpha_a|\alpha_b)\big)} \quad \text{for} \quad a \in \{1, 2, \dots, r\}. \tag{2.8}$$

Here, $(\alpha_a|\alpha_b) = 2\delta_{a,b} - \delta_{a,b+1} - \delta_{a,b-1}$ is the Cartan matrix of $sl(r+1)$; $\big\{b^{(a)}_j\big\}$ are the Dynkin labels which characterize the quantum space: $\big[b^{(1)}_{2j}, b^{(2)}_{2j}, \dots, b^{(r)}_{2j}\big] = [0, 0, \dots, 1]$, $\big[b^{(1)}_{2j-1}, b^{(2)}_{2j-1}, \dots, b^{(r)}_{2j-1}\big] = [1, 0, \dots, 0]$; $\big\{w^{(a)}_j\big\}$ are inhomogeneity parameters: $w^{(a)}_{2j-1} = \delta_{a,1}\mathrm{i}u$, $w^{(a)}_{2j} = -\delta_{a,r}\mathrm{i}\big(u + \frac{1}{2}g\big)$; $g = r+1$ (the dual Coxeter number); $\zeta_a = \mathrm{e}^{-\frac{\mu_a - \mu_{a+1}}{T}}$. Note that the Dynkin labels for the even and odd sites are those for conjugate representations each other. The non-negative integers $\{M_a\}$ should satisfy the condition, $\sum_{j=1}^{N} b^{(a)}_j - \sum_{b=1}^{r} M_b(\alpha_b|\alpha_a) = \frac{N}{2}(\delta_{a,1} + \delta_{a,r}) + M_{a-1} - 2M_a + M_{a+1} \in \mathbb{Z}_{\geqslant 0}$, where $M_0 = M_{r+1} = 0$. For the row-to-row transfer matrix, the above parameters take the values: $\big[b^{(1)}_j, b^{(2)}_j, \dots, b^{(r)}_j\big] = [1, 0, \dots, 0]$; $w^{(a)}_j = 0$; $\zeta_a = 1$.

For $a \in \{1, 2, \dots, r\}$ and $m \in \mathbb{Z}_{\geqslant 1}$, we normalize equation (2.4) as $\widetilde{T}^{(a)}_m(v) = T^{(a)}_m(v) \big/ \widetilde{\mathcal{N}}^{(a)}_m(v)$, where

$$\widetilde{\mathcal{N}}^{(a)}_m(v) = \frac{\phi_-\big(v - \frac{a+m}{2}\mathrm{i}\big)\phi_+\big(v + \frac{a+m}{2}\mathrm{i}\big)}{\phi_-\big(v - \frac{a-m}{2}\mathrm{i}\big)\phi_+\big(v + \frac{a-m}{2}\mathrm{i}\big)} \prod_{j=1}^{a}\prod_{k=1}^{m} \phi_-\left(v - \frac{a-m-2j+2k}{2}\mathrm{i}\right)$$
$$\times \phi_+\left(v - \frac{a-m-2j+2k}{2}\mathrm{i}\right). \tag{2.9}$$

Note that the poles of $\widetilde{T}^{(a)}_m(v)$ from the functions $\{Q_b(v)\}$ (dress part) are spurious under the BAE (2.7). We can show that $\widetilde{T}^{(a)}_m(v)$ satisfies the following functional relation

$$\widetilde{T}^{(a)}_m\left(v+\frac{\mathrm{i}}{2}\right)\widetilde{T}^{(a)}_m\left(v-\frac{\mathrm{i}}{2}\right)=\widetilde{T}^{(a)}_{m+1}(v)\widetilde{T}^{(a)}_{m-1}(v)+\widetilde{T}^{(a-1)}_m(v)\widetilde{T}^{(a+1)}_m(v)$$
$$\text{for}\quad a\in\{1,2,\ldots,r\}\quad\text{and}\quad m\in\mathbb{Z}_{\geqslant 1} \tag{2.10}$$

where

$$\widetilde{T}^{(a)}_0(v)=1\quad\text{for}\quad a\in\mathbb{Z}_{\geqslant 1}$$
$$\widetilde{T}^{(0)}_m(v)=\frac{\phi_-\left(v+\frac{m}{2}\mathrm{i}\right)\phi_+\left(v-\frac{m}{2}\mathrm{i}\right)}{\phi_-\left(v-\frac{m}{2}\mathrm{i}\right)\phi_+\left(v+\frac{m}{2}\mathrm{i}\right)}\quad\text{for}\quad m\in\mathbb{Z}_{\geqslant 1} \tag{2.11}$$
$$\widetilde{T}^{(r+1)}_m(v)=\mathrm{e}^{\frac{m(\mu_1+\mu_2+\cdots+\mu_{r+1})}{T}}\quad\text{for}\quad m\in\mathbb{Z}_{\geqslant 1}.$$

Except for the vacuum part (the part made from the function (2.6)), this equation has the same form as the $sl(r+1)$ T-system in [21].

3. Traditional TBA equations

In this section, we transform the T-system (2.10) into the traditional TBA equations by the standard procedure [20, 4]. A similar argument in relation to Stokes multipliers can be found in section 5 in [33].

For $m\in\mathbb{Z}_{\geqslant 1}$ and $a\in\{1,2,\ldots,r\}$, we define functions:

$$Y^{(a)}_m(v)=\frac{\widetilde{T}^{(a)}_{m+1}(v)\widetilde{T}^{(a)}_{m-1}(v)}{\widetilde{T}^{(a-1)}_m(v)\widetilde{T}^{(a+1)}_m(v)}=\frac{T^{(a)}_{m+1}(v)T^{(a)}_{m-1}(v)}{T^{(a-1)}_m(v)T^{(a+1)}_m(v)}. \tag{3.1}$$

By using the T-system (2.10), we can show that equations (3.1) satisfy the following Y-system

$$Y^{(a)}_m\left(v+\frac{\mathrm{i}}{2}\right)Y^{(a)}_m\left(v-\frac{\mathrm{i}}{2}\right)=\frac{\left(1+Y^{(a)}_{m+1}(v)\right)\left(1+Y^{(a)}_{m-1}(v)\right)}{\prod_{b=1}^{r}\left(1+\left(Y^{(b)}_m(v)\right)^{-1}\right)^{I_{ab}}}$$
$$\text{for}\quad a\in\{1,2,\ldots,r\}\quad\text{and}\quad m\in\mathbb{Z}_{\geqslant 1} \tag{3.2}$$

where $I_{ab}=\delta_{a,b+1}+\delta_{a,b-1}$, $Y^{(a)}_0(v)=0$.

From a numerical analysis for finite N, u, r, we expect that a one-string solution (for every colour) in the sector $\frac{N}{2}=M_1=M_2=\cdots=M_r$ of the BAE (2.7) provides the largest eigenvalue of the QTM (2.2) at $v=0$ at least for the case, $\mu_1=\mu_2=\cdots=\mu_{r+1}=0$ (cf figure 1). Hereafter we consider only this one-string solution. The following conjecture will be valid for this one-string solution (cf figures 2 and 3).

Conjecture 3.1. *For small u ($|u|\ll 1$), $a\in\{1,2,\ldots,r\}$ and $m\in\mathbb{Z}_{\geqslant 1}$, every zero $\left\{\tilde{z}^{(a)}_m\right\}$ of $\widetilde{T}^{(a)}_m(v)$ is located near the lines $\operatorname{Im} v=\pm\frac{m+a}{2}$ at least for the case, $\mu_1=\mu_2=\cdots=\mu_{r+1}=0$.*

Based on this conjecture, we establish the ANZC (analytic, nonzero and constant asymptotics in the limit $|v|\to\infty$) property in some domain for the functions (3.1). We find that $Y^{(a)}_m(v)$ has constant asymptotics

$$\lim_{|v|\to\infty}Y^{(a)}_m(v)=\frac{Q^{(a)}_{m-1}Q^{(a)}_{m+1}}{Q^{(a-1)}_m Q^{(a+1)}_m} \tag{3.3}$$

where $\left\{Q^{(a)}_m\right\}$ is given by equation (2.4) with a formal setting $Q_a(v)=\phi_\pm(v)=1$. $\left\{Q^{(a)}_m\right\}$ is also characterized as a solution of the Q-system [28]

$$\left(Q^{(a)}_m\right)^2=Q^{(a)}_{m-1}Q^{(a)}_{m+1}+Q^{(a-1)}_m Q^{(a+1)}_m\quad\text{for}\quad a\in\{1,2,\ldots,r\}\quad\text{and}\quad m\in\mathbb{Z}_{\geqslant 1} \tag{3.4}$$

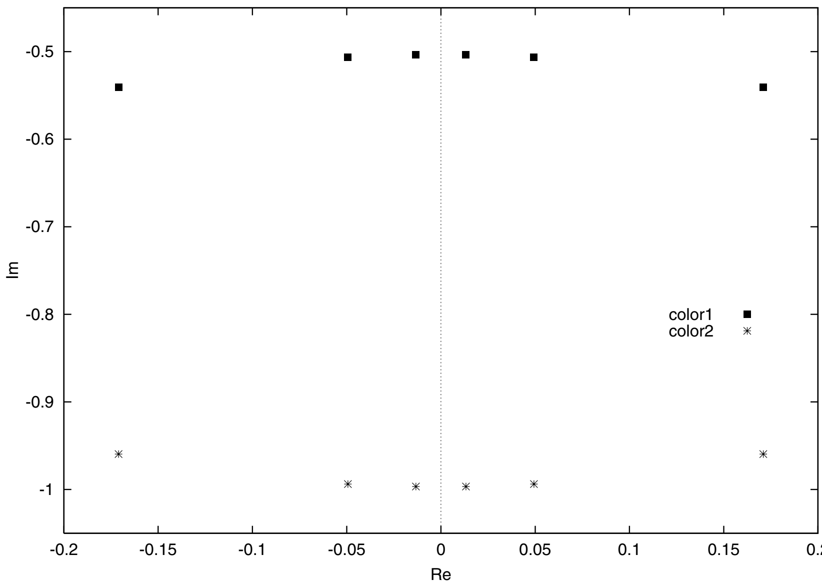

Figure 1. Location of the roots of the BAE for the $sl(3)$ case ($N = 12, u = -0.05, \mu_1 = \mu_2 = \mu_3 = 0$), which gives the largest eigenvalue of the QTM $t_1^{(1)}(v)$ at $v = 0$. Both colour one roots $\{v_k^{(1)}\}$ and colour two roots $\{v_k^{(2)}\}$ form six one-strings.

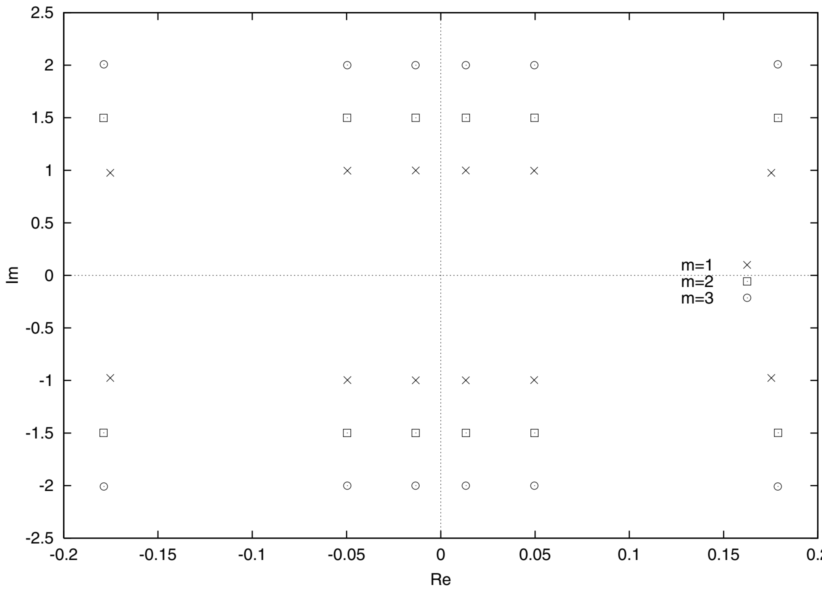

Figure 2. Location of zeros of $\widetilde{T}_m^{(1)}(v)$ for the $sl(3)$ case ($m = 1, 2, 3, N = 12, u = -0.05, \mu_1 = \mu_2 = \mu_3 = 0$). The zeros recede from the physical strip $\mathrm{Im}\, v \in [-\frac{1}{2}, \frac{1}{2}]$ as m increases.

where $Q_0^{(a)} = Q_m^{(0)} = 1$; $Q_m^{(r+1)} = \mathrm{e}^{\frac{m(\mu_1+\mu_2+\cdots+\mu_{r+1})}{T}}$. Note that $Q_m^{(a)}$ coincides with the character of mth symmetric ath anti-symmetric tensor representation $V(m\Lambda_a)$ of $gl(r+1)$ if we set $\frac{\mu_a}{T} \to \epsilon_a$, where $\{\epsilon_a\}$ are orthonormal bases of the dual space of the Cartan subalgebra.

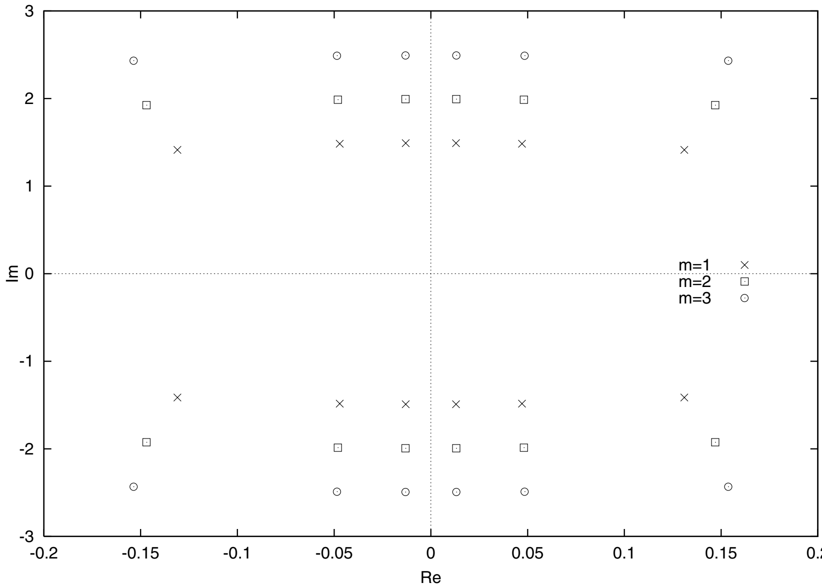

Figure 3. Location of zeros of $\widetilde{T}^{(2)}_m(v)$ for the $sl(3)$ case ($m = 1, 2, 3$, $N = 12$, $u = -0.05$, $\mu_1 = \mu_2 = \mu_3 = 0$). The zeros recede from the physical strip $\mathrm{Im}\, v \in [-\frac{1}{2}, \frac{1}{2}]$ as m increases.

In particular, for the $\mu_1 = \mu_2 = \cdots = \mu_{r+1} = 0$ case, $Q^{(a)}_m$ is given by

$$Q^{(a)}_m = \left(\frac{(m+g)!m!}{(m+a)!(m+g-a)!} \right)^m \prod_{k=1}^{m} \left\{ \frac{(k+a)(k+g-a)}{k(k+g)} \right\}^k \tag{3.5}$$

and equation (3.3) has the form

$$\lim_{|v|\to\infty} Y^{(a)}_m(v) = \frac{m(g+m)}{a(g-a)}. \tag{3.6}$$

Equation (3.5) is the dimension of $V(m\Lambda_a)$. Based on the conjecture 3.1 and equation (3.3), we can see that the functions $1 + Y^{(a)}_m(v)$, $1 + \left(Y^{(a)}_m(v)\right)^{-1}$ in the domain $\mathrm{Im}\, v \in [-\delta, \delta]$ $(0 < \delta \ll 1)$ and $Y^{(a)}_m(v)$ for $(a, m) \neq (1, 1)$ in the domain $\mathrm{Im}\, v \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ (physical strip) have the ANZC property at least for the case $\mu_1 = \cdots = \mu_{r+1} = 0$. However, $Y^{(1)}_1(v)$ has zeros of order $N/2$ at $\pm\mathrm{i}\left(\frac{1}{2} + u\right)$ if $u < 0$ $(J > 0)$, poles of order $N/2$ at $\pm\mathrm{i}\left(\frac{1}{2} - u\right)$ if $u > 0$ $(J < 0)$ in the physical strip. Thus, we should modify $Y^{(a)}_m(v)$ as

$$\widetilde{Y}^{(a)}_m(v) = Y^{(a)}_m(v) \left\{ \tanh\frac{\pi}{2}\left(v + \mathrm{i}\left(\frac{1}{2} \pm u\right)\right) \tanh\frac{\pi}{2}\left(v - \mathrm{i}\left(\frac{1}{2} \pm u\right)\right) \right\}^{\mp\frac{N\delta_{a,1}\delta_{m,1}}{2}} \tag{3.7}$$

where the sign $\pm$ is equal to that of $-u$. By using the relation

$$\tanh\frac{\pi}{4}(v + \mathrm{i}) \tanh\frac{\pi}{4}(v - \mathrm{i}) = 1 \tag{3.8}$$

we modify the left-hand side (lhs) of the Y-system (3.2) as

$$\widetilde{Y}^{(a)}_m\left(v - \frac{\mathrm{i}}{2}\right) \widetilde{Y}^{(a)}_m\left(v + \frac{\mathrm{i}}{2}\right) = \frac{\left(1 + Y^{(a)}_{m+1}(v)\right)\left(1 + Y^{(a)}_{m-1}(v)\right)}{\prod_{b=1}^{r} \left(1 + \left(Y^{(b)}_m(v)\right)^{-1}\right)^{I_{ab}}}$$
$$\text{for} \quad m \in \mathbb{Z}_{\geqslant 1} \quad \text{and} \quad a \in \{1, 2, \ldots, r\}. \tag{3.9}$$

This modified Y-system has the ANZC property. Then we can transform equation (3.9) into nonlinear integral equations by a standard procedure:

$$\log Y_m^{(a)}(v) = \pm\frac{N\delta_{a1}\delta_{m1}}{2}\log\left\{\tanh\frac{\pi}{2}\left(v+\mathrm{i}\left(\frac{1}{2}\pm u\right)\right)\tanh\frac{\pi}{2}\left(v-\mathrm{i}\left(\frac{1}{2}\pm u\right)\right)\right\}$$
$$+K*\log\left\{\frac{\left(1+Y_{m-1}^{(a)}\right)\left(1+Y_{m+1}^{(a)}\right)}{\prod_{b=1}^{r}\left(1+\left(Y_m^{(b)}\right)^{-1}\right)^{I_{ab}}}\right\}(v)$$
$$\text{for}\quad a\in\{1,2,\ldots,r\}\quad\text{and}\quad m\in\mathbb{Z}_{\geqslant 1} \tag{3.10}$$

where $Y_0^{(a)}(v) = 0$ and $*$ is the convolution. After the Trotter limit $N\to\infty$ with $u = -\frac{J}{NT}$, we obtain a TBA equation

$$\log Y_m^{(a)}(v) = -\frac{\pi J\delta_{a1}\delta_{m1}}{T\cosh\pi v}+K*\log\left\{\frac{\left(1+Y_{m-1}^{(a)}\right)\left(1+Y_{m+1}^{(a)}\right)}{\prod_{b=1}^{r}\left(1+\left(Y_m^{(b)}\right)^{-1}\right)^{I_{ab}}}\right\}(v)$$
$$\text{for}\quad a\in\{1,2,\ldots,r\}\quad\text{and}\quad m\in\mathbb{Z}_{\geqslant 1} \tag{3.11}$$

where $Y_0^{(a)}(v) := 0$ and the kernel is

$$K(v) = \frac{1}{2\cosh\pi v}. \tag{3.12}$$

We can also rewrite equation (3.11) as

$$-\frac{\pi J\delta_{a1}\delta_{m1}}{T\cosh\pi v} = \sum_{n=1}^{\infty}K^{m,n}*\log\left(1+Y_n^{(a)}\right)(v)-\sum_{b=1}^{r}\sum_{n=1}^{\infty}J_{a,b}^{m,n}*\log\left(1+\left(Y_n^{(b)}\right)^{-1}\right)(v) \tag{3.13}$$

where

$$\begin{aligned}K^{m,n}(v) &= \delta_{m,n}\delta(v)-(\delta_{m,n-1}+\delta_{m,n+1})K(v)\\ J_{a,b}^{n,k}(v) &= (\delta_{a,b}\delta(v)-(\delta_{a,b-1}+\delta_{a,b+1})K(v))\delta_{n,k}.\end{aligned} \tag{3.14}$$

We find that the TBA equation (3.13) coincides with that from the string hypothesis in [29] after suitable modification. By using the following relations

$$\begin{aligned}A^{a,d}(v) &= \sum_{l=1}^{\min(a,d)}G_{|a-d|+2l-1}(v)\\ G_a(v) &= \frac{1}{g}\frac{\sin\frac{(g-a)\pi}{g}}{\cos\frac{(g-a)\pi}{g}+\cosh\frac{2\pi v}{g}}\\ \widehat{A}^{a,d}(k) &= \int_{-\infty}^{\infty}\mathrm{d}v A^{a,d}(v)\,\mathrm{e}^{-\mathrm{i}kv} = \frac{\sinh\left(\frac{\min(a,d)}{2}k\right)\sinh\left(\frac{g-\min(a,d)}{2}k\right)}{\sinh\left(\frac{k}{2}\right)\sinh\left(\frac{g}{2}k\right)}\\ \sum_{l=1}^{r}\widehat{A}^{m,l}(k)\widehat{K}^{l,n}(k) &= \frac{\delta_{m,n}}{2\cosh\left(\frac{k}{2}\right)}\\ \widehat{K}^{m,n}(k) &= \delta_{m,n}-\frac{\delta_{m,n-1}+\delta_{m,n+1}}{2\cosh\frac{k}{2}}\end{aligned} \tag{3.15}$$

we can also rewrite the TBA equation (3.11) as

$$\log Y_m^{(a)}(v) = -\frac{2\pi J\delta_{m,1}}{T}G_a(v)+\sum_{b=1}^{r}A^{a,b}*\log\left\{\frac{\left(1+Y_{m-1}^{(b)}\right)\left(1+Y_{m+1}^{(b)}\right)}{\prod_{d=1}^{r}\left(1+Y_m^{(d)}\right)^{I_{bd}}}\right\}(v)$$
$$\text{for}\quad a\in\{1,2,\ldots,r\}\quad\text{and}\quad m\in\mathbb{Z}_{\geqslant 1} \tag{3.16}$$

where $Y_0^{(a)}(v) = 0$. This TBA equation (3.16) does not contain $(Y_m^{(a)})^{-1}$, which contrasts with equations (3.11) and (3.13). We can derive the following relation from equations (2.10), (3.1) and (3.15):

$$\log \widetilde{T}_1^{(1)}(v) = \sum_{b=1}^{r} G_b * \log\left(1 + Y_1^{(b)}\right)(v) - 2N \int_0^{\infty} \mathrm{d}k \frac{\mathrm{e}^{-\frac{1}{2}k} \sinh(ku)\cos(kv)\sinh\left(\frac{g-1}{2}k\right)}{k \sinh\left(\frac{g}{2}k\right)}. \tag{3.17}$$

After the Trotter limit $N \to \infty$ with $u = -\frac{J}{NT}$, we have the free energy per site:

$$\begin{aligned} f &= -T \lim_{N\to\infty} \log T_1^{(1)}(0) = J - T \lim_{N\to\infty} \log \widetilde{T}_1^{(1)}(0) \\ &= J\left\{\frac{2}{g}\left(\gamma + \psi\left(\frac{1}{g}\right)\right) + 1\right\} - T\sum_{a=1}^{r}\int_{-\infty}^{\infty} \mathrm{d}v G_a(v) \log\left(1 + Y_1^{(a)}(v)\right) \end{aligned} \tag{3.18}$$

where $\psi(z)$ is the digamma function

$$\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z} \log\Gamma(z) \qquad \gamma = -\psi(1). \tag{3.19}$$

4. New nonlinear integral equations

It has been pointed out [13] that Takahashi's NLIE for the XXZ-model [19] can be rederived from the T-system of the QTM. In this section, we derive our new NLIEs (4.6) and (4.7) from the T-system (2.10).

$\widetilde{T}_m^{(a)}(v)$ has poles only at $\pm\tilde{\beta}_m^{(a)}$: $\tilde{\beta}_m^{(a)} = \frac{m+a}{2}\mathrm{i} + \mathrm{i}u$. These poles are of order $N/2$ at most. In addition, $\lim_{|v|\to\infty} \widetilde{T}_m^{(a)}(v) = Q_m^{(a)}$ is a finite number (cf equation (3.5)). Thus we must put

$$\widetilde{T}_m^{(a)}(v) = Q_m^{(a)} + \sum_{j=1}^{\frac{N}{2}} \left\{ \frac{b_{m,j}^{(a)}}{\left(v - \tilde{\beta}_m^{(a)}\right)^j} + \frac{\overline{b}_{m,j}^{(a)}}{\left(v + \tilde{\beta}_m^{(a)}\right)^j} \right\} \tag{4.1}$$

where the coefficients $b_{m,j}^{(a)}, \overline{b}_{m,j}^{(a)} \in \mathbb{C}$ are given as follows:

$$b_{m,j}^{(a)} = \oint_{C_m^{(a)}} \frac{\mathrm{d}v}{2\pi\mathrm{i}} \widetilde{T}_m^{(a)}(v)\left(v - \tilde{\beta}_m^{(a)}\right)^{j-1} \qquad \overline{b}_{m,j}^{(a)} = \oint_{\overline{C}_m^{(a)}} \frac{\mathrm{d}v}{2\pi\mathrm{i}} \widetilde{T}_m^{(a)}(v)\left(v + \tilde{\beta}_m^{(a)}\right)^{j-1}. \tag{4.2}$$

Here the contour $C_m^{(a)}$ is a counterclockwise closed loop around $\tilde{\beta}_m^{(a)}$ which does not surround $-\tilde{\beta}_m^{(a)}$; the contour $\overline{C}_m^{(a)}$ is a counterclockwise closed loop around $-\tilde{\beta}_m^{(a)}$ which does not surround $\tilde{\beta}_m^{(a)}$. Using the T-system (2.10), we can modify (4.2) as

$$\begin{aligned} b_{m,j}^{(a)} &= \oint_{C_m^{(a)}} \frac{\mathrm{d}v}{2\pi\mathrm{i}} \left\{ \frac{\widetilde{T}_{m-1}^{(a)}\left(v - \frac{\mathrm{i}}{2}\right)\widetilde{T}_{m+1}^{(a)}\left(v - \frac{\mathrm{i}}{2}\right)}{\widetilde{T}_m^{(a)}(v - \mathrm{i})} + \frac{\widetilde{T}_m^{(a-1)}\left(v - \frac{\mathrm{i}}{2}\right)\widetilde{T}_m^{(a+1)}\left(v - \frac{\mathrm{i}}{2}\right)}{\widetilde{T}_m^{(a)}(v - \mathrm{i})} \right\} \left(v - \tilde{\beta}_m^{(a)}\right)^{j-1} \\ \overline{b}_{m,j}^{(a)} &= \oint_{\overline{C}_m^{(a)}} \frac{\mathrm{d}v}{2\pi\mathrm{i}} \left\{ \frac{\widetilde{T}_{m-1}^{(a)}\left(v + \frac{\mathrm{i}}{2}\right)\widetilde{T}_{m+1}^{(a)}\left(v + \frac{\mathrm{i}}{2}\right)}{\widetilde{T}_m^{(a)}(v + \mathrm{i})} + \frac{\widetilde{T}_m^{(a-1)}\left(v + \frac{\mathrm{i}}{2}\right)\widetilde{T}_m^{(a+1)}\left(v + \frac{\mathrm{i}}{2}\right)}{\widetilde{T}_m^{(a)}(v + \mathrm{i})} \right\} \left(v + \tilde{\beta}_m^{(a)}\right)^{j-1}. \end{aligned} \tag{4.3}$$

Substituting equation (4.3) into equation (4.1), we obtain

$$\widetilde{T}^{(a)}_m(v) = Q^{(a)}_m + \oint_{C^{(a)}_m} \frac{\mathrm{d}y}{2\pi\mathrm{i}} \frac{1-\left(\frac{y}{v-\tilde{\beta}^{(a)}_m}\right)^{\frac{N}{2}}}{v-y-\tilde{\beta}^{(a)}_m} \left\{ \frac{\widetilde{T}^{(a)}_{m-1}\left(y+\tilde{\beta}^{(a)}_m-\frac{\mathrm{i}}{2}\right)\widetilde{T}^{(a)}_{m+1}\left(y+\tilde{\beta}^{(a)}_m-\frac{\mathrm{i}}{2}\right)}{\widetilde{T}^{(a)}_m\left(y+\tilde{\beta}^{(a)}_m-\mathrm{i}\right)} \right.$$
$$\left. + \frac{\widetilde{T}^{(a-1)}_m\left(y+\tilde{\beta}^{(a)}_m-\frac{\mathrm{i}}{2}\right)\widetilde{T}^{(a+1)}_m\left(y+\tilde{\beta}^{(a)}_m-\frac{\mathrm{i}}{2}\right)}{\widetilde{T}^{(a)}_m\left(y+\tilde{\beta}^{(a)}_m-\mathrm{i}\right)} \right\}$$
$$+ \oint_{\overline{C}^{(a)}_m} \frac{\mathrm{d}y}{2\pi\mathrm{i}} \frac{1-\left(\frac{y}{v+\tilde{\beta}^{(a)}_m}\right)^{\frac{N}{2}}}{v-y+\tilde{\beta}^{(a)}_m} \left\{ \frac{\widetilde{T}^{(a)}_{m-1}\left(y-\tilde{\beta}^{(a)}_m+\frac{\mathrm{i}}{2}\right)\widetilde{T}^{(a)}_{m+1}\left(y-\tilde{\beta}^{(a)}_m+\frac{\mathrm{i}}{2}\right)}{\widetilde{T}^{(a)}_m\left(y-\tilde{\beta}^{(a)}_m+\mathrm{i}\right)} \right.$$
$$\left. + \frac{\widetilde{T}^{(a-1)}_m\left(y-\tilde{\beta}^{(a)}_m+\frac{\mathrm{i}}{2}\right)\widetilde{T}^{(a+1)}_m\left(y-\tilde{\beta}^{(a)}_m+\frac{\mathrm{i}}{2}\right)}{\widetilde{T}^{(a)}_m\left(y-\tilde{\beta}^{(a)}_m+\mathrm{i}\right)} \right\}$$
$$\text{for} \quad a \in \{1,2,\ldots,r\} \quad \text{and} \quad m \in \mathbb{Z}_{\geqslant 1} \tag{4.4}$$

where the contour $C^{(a)}_m$ $\left(\overline{C}^{(a)}_m\right)$ is a counterclockwise closed loop around 0 which does not surround $-2\tilde{\beta}^{(a)}_m$ $\left(2\tilde{\beta}^{(a)}_m\right)$. The first and second terms in the first bracket $\{\cdots\}$ in equation (4.4) have a common singularity at 0; while for $m=1$, this singularity from the first term disappears since $\widetilde{T}^{(a)}_0(y)=1$. Thus for $m=1$, the contribution to the contour integral from the first term in the first bracket $\{\cdots\}$ in equation (4.4) vanishes if the contour $C^{(a)}_1$ does not surround the singularity at $\tilde{z}^{(a)}_1-\tilde{\beta}^{(a)}_1+\mathrm{i}$ (cf conjecture 3.1). This is also the case with the second bracket $\{\cdots\}$ in equation (4.4). Therefore for $m=1$, equation (4.4) reduces to

$$\widetilde{T}^{(a)}_1(v) = Q^{(a)}_1 + \oint_{C^{(a)}_1} \frac{\mathrm{d}y}{2\pi\mathrm{i}} \frac{1-\left(\frac{y}{v-\tilde{\beta}^{(a)}_1}\right)^{\frac{N}{2}}}{v-y-\tilde{\beta}^{(a)}_1} \frac{\widetilde{T}^{(a-1)}_1\left(y+\tilde{\beta}^{(a)}_1-\frac{\mathrm{i}}{2}\right)\widetilde{T}^{(a+1)}_1\left(y+\tilde{\beta}^{(a)}_1-\frac{\mathrm{i}}{2}\right)}{\widetilde{T}^{(a)}_1\left(y+\tilde{\beta}^{(a)}_1-\mathrm{i}\right)}$$
$$+ \oint_{\overline{C}^{(a)}_1} \frac{\mathrm{d}y}{2\pi\mathrm{i}} \frac{1-\left(\frac{y}{v+\tilde{\beta}^{(a)}_1}\right)^{\frac{N}{2}}}{v-y+\tilde{\beta}^{(a)}_1} \frac{\widetilde{T}^{(a-1)}_1\left(y-\tilde{\beta}^{(a)}_1+\frac{\mathrm{i}}{2}\right)\widetilde{T}^{(a+1)}_1\left(y-\tilde{\beta}^{(a)}_1+\frac{\mathrm{i}}{2}\right)}{\widetilde{T}^{(a)}_1\left(y-\tilde{\beta}^{(a)}_1+\mathrm{i}\right)}$$
$$\text{for} \quad a \in \{1,2,\ldots,r\} \tag{4.5}$$

where the contour $C^{(a)}_1$ $\left(\overline{C}^{(a)}_1\right)$ is a counterclockwise closed loop around 0 which does not surround $\tilde{z}^{(a)}_1-\tilde{\beta}^{(a)}_1+\mathrm{i}, -2\tilde{\beta}^{(a)}_1$ $\left(\tilde{z}^{(a)}_1+\tilde{\beta}^{(a)}_1-\mathrm{i}, 2\tilde{\beta}^{(a)}_1\right)$. Now we take the Trotter limit $N\to\infty$ in equation (4.4):

$$\mathcal{T}^{(a)}_m(v) = Q^{(a)}_m + \oint_{C^{(a)}_m} \frac{\mathrm{d}y}{2\pi\mathrm{i}} \frac{1}{v-y-\beta^{(a)}_m} \left\{ \frac{\mathcal{T}^{(a)}_{m-1}\left(y+\beta^{(a)}_m-\frac{\mathrm{i}}{2}\right)\mathcal{T}^{(a)}_{m+1}\left(y+\beta^{(a)}_m-\frac{\mathrm{i}}{2}\right)}{\mathcal{T}^{(a)}_m\left(y+\beta^{(a)}_m-\mathrm{i}\right)} \right.$$
$$\left. + \frac{\mathcal{T}^{(a-1)}_m\left(y+\beta^{(a)}_m-\frac{\mathrm{i}}{2}\right)\mathcal{T}^{(a+1)}_m\left(y+\beta^{(a)}_m-\frac{\mathrm{i}}{2}\right)}{\mathcal{T}^{(a)}_m\left(y+\beta^{(a)}_m-\mathrm{i}\right)} \right\}$$
$$+ \oint_{\overline{C}^{(a)}_m} \frac{\mathrm{d}y}{2\pi\mathrm{i}} \frac{1}{v-y+\beta^{(a)}_m} \left\{ \frac{\mathcal{T}^{(a)}_{m-1}\left(y-\beta^{(a)}_m+\frac{\mathrm{i}}{2}\right)\mathcal{T}^{(a)}_{m+1}\left(y-\beta^{(a)}_m+\frac{\mathrm{i}}{2}\right)}{\mathcal{T}^{(a)}_m\left(y-\beta^{(a)}_m+\mathrm{i}\right)} \right.$$
$$\left. + \frac{\mathcal{T}^{(a-1)}_m\left(y-\beta^{(a)}_m+\frac{\mathrm{i}}{2}\right)\mathcal{T}^{(a+1)}_m\left(y-\beta^{(a)}_m+\frac{\mathrm{i}}{2}\right)}{\mathcal{T}^{(a)}_m\left(y-\beta^{(a)}_m+\mathrm{i}\right)} \right\}$$
$$\text{for} \quad a \in \{1,2,\ldots,r\} \quad \text{and} \quad m \in \mathbb{Z}_{\geqslant 1}. \tag{4.6}$$

Here $\mathcal{T}_m^{(a)}(v) := \lim_{N\to\infty} \widetilde{T}_m^{(a)}(v)$; $\beta_m^{(a)} := \lim_{N\to\infty} \tilde{\beta}_m^{(a)} = \frac{m+a}{2}\mathrm{i}$; $\mathcal{T}_m^{(r+1)}(v) = \mathrm{e}^{\frac{m(\mu_1+\cdots+\mu_{r+1})}{T}}$; $\mathcal{T}_0^{(a)}(v) = 1$; $\mathcal{T}_m^{(0)}(v) = \exp\left(\frac{mJ}{(v^2+\frac{m^2}{4})T}\right)$; the contour $C_m^{(a)}$ $\left(\overline{C}_m^{(a)}\right)$ is a counterclockwise closed loop around 0 which satisfies the condition $y \neq v - \beta_m^{(a)}$ $\left(y \neq v + \beta_m^{(a)}\right)$ and does not surround $-2\beta_m^{(a)}$ $\left(2\beta_m^{(a)}\right)$. In particular for $m = 1$, we obtain a system of NLIEs, which contains only a *finite* number of unknown functions $\left\{\mathcal{T}_1^{(a)}(v)\right\}_{1\leqslant a\leqslant r}$:

$$\begin{aligned}\mathcal{T}_1^{(a)}(v) = Q_1^{(a)} &+ \oint_{C_1^{(a)}} \frac{\mathrm{d}y}{2\pi\mathrm{i}} \frac{\mathcal{T}_1^{(a-1)}\left(y + \beta_1^{(a)} - \frac{\mathrm{i}}{2}\right)\mathcal{T}_1^{(a+1)}\left(y + \beta_1^{(a)} - \frac{\mathrm{i}}{2}\right)}{\left(v - y - \beta_1^{(a)}\right)\mathcal{T}_1^{(a)}\left(y + \beta_1^{(a)} - \mathrm{i}\right)} \\ &+ \oint_{\overline{C}_1^{(a)}} \frac{\mathrm{d}y}{2\pi\mathrm{i}} \frac{\mathcal{T}_1^{(a-1)}\left(y - \beta_1^{(a)} + \frac{\mathrm{i}}{2}\right)\mathcal{T}_1^{(a+1)}\left(y - \beta_1^{(a)} + \frac{\mathrm{i}}{2}\right)}{\left(v - y + \beta_1^{(a)}\right)\mathcal{T}_1^{(a)}\left(y - \beta_1^{(a)} + \mathrm{i}\right)} \\ &\text{for} \quad a \in \{1, 2, \ldots, r\}. \end{aligned} \tag{4.7}$$

Here $\mathcal{T}_1^{(r+1)}(v) = \mathrm{e}^{\frac{\mu_1+\cdots+\mu_{r+1}}{T}}$; the contour $C_1^{(a)}$ $\left(\overline{C}_1^{(a)}\right)$ is a counterclockwise closed loop around 0 which satisfies the condition $y \neq v - \beta_1^{(a)}$ $\left(y \neq v + \beta_1^{(a)}\right)$ and does not surround $z_1^{(a)} - \beta_1^{(a)} + \mathrm{i}, -2\beta_1^{(a)}$ $\left(z_1^{(a)} + \beta_1^{(a)} - \mathrm{i}, 2\beta_1^{(a)}\right)$; $z_1^{(a)} = \lim_{N\to\infty} \tilde{z}_1^{(a)}$. We can calculate the free energy per site f by using equation (4.7) and the relation

$$f = J - T \log \mathcal{T}_1^{(1)}(0). \tag{4.8}$$

We need not use equation (4.6) (for $m \in \mathbb{Z}_{\geqslant 2}$) to obtain the free energy per site. In addition, $\left\{\mathcal{T}_m^{(a)}(v)\right\}_{1\leqslant a\leqslant r; m\in\mathbb{Z}_{\geqslant 1}}$ is given in terms of the solution of equation (4.7) $\left\{\mathcal{T}_1^{(a)}(v)\right\}_{1\leqslant a\leqslant r}$ based on a Jacobi–Trudi formula [29]. However, equation (4.6) has theoretical significance since it manifests the relation between our new NLIEs and the traditional TBA equations (3.11), (3.13) and (3.16). In fact, equations (4.6) and (4.7) are related to the traditional TBA equations (3.11), (3.13) and, (3.16) through the relation (3.1) in the Trotter limit.

5. High-temperature expansion

In this section, we calculate the high-temperature expansion of the free energy (4.8) for $r = 1, 2, 3$ using our new NLIEs (4.7). As for the XXX-model case, the high-temperature expansion of the free energy is calculated [30] by Takahashi's NLIE up to order 100. We assume the following expansion for large T:

$$\mathcal{T}_1^{(a)}(v) = \exp\left(\sum_{n=0}^{\infty} b_n^{(a)}(v) \left(\frac{J}{T}\right)^n\right). \tag{5.1}$$

In contrast with the XXX-model case [30], we need to further assume

$$b_n^{(a)}(v) = \sum_{j=0}^{n-1} \frac{c_{n,j}^{(a)} v^{2j}}{\left(v^2 + \frac{(a+1)^2}{4}\right)^n} \tag{5.2}$$

where $c_{n,j}^{(a)} \in \mathbb{C}$. Substituting equation (5.1) into equation (4.7), we can obtain the coefficients $\{b_n^{(a)}(v)\}$. Here we just enumerate the first few of them. $b_0^{(a)}(v)$ has the form $b_0^{(a)}(v) = \log Q_1^{(a)}$.

Other coefficients are given as follows:
sl(2) case

$$
\begin{aligned}
c_{1,0}^{(1)} &= \frac{2Q_1^{(2)}}{Q_1^{(1)2}} \\
c_{2,0}^{(1)} &= \frac{3Q_1^{(2)}}{Q_1^{(1)2}} - \frac{6Q_1^{(2)2}}{Q_1^{(1)4}} \\
c_{2,1}^{(1)} &= \frac{Q_1^{(2)}}{Q_1^{(1)2}} - \frac{4Q_1^{(2)2}}{Q_1^{(1)4}} \\
c_{3,0}^{(1)} &= \frac{10Q_1^{(2)}}{3Q_1^{(1)2}} - \frac{18Q_1^{(2)2}}{Q_1^{(1)4}} + \frac{80Q_1^{(2)3}}{3Q_1^{(1)6}} \\
c_{3,1}^{(1)} &= \frac{3Q_1^{(2)}}{Q_1^{(1)2}} - \frac{22Q_1^{(2)2}}{Q_1^{(1)4}} + \frac{40Q_1^{(2)3}}{Q_1^{(1)6}} \\
c_{3,2}^{(1)} &= \frac{Q_1^{(2)}}{Q_1^{(1)2}} - \frac{8Q_1^{(2)2}}{Q_1^{(1)4}} + \frac{16Q_1^{(2)3}}{Q_1^{(1)6}}.
\end{aligned}
\tag{5.3}
$$

sl(3) case

$$
\begin{aligned}
c_{1,0}^{(1)} &= \frac{2Q_1^{(2)}}{Q_1^{(1)2}} \\
c_{2,0}^{(1)} &= \frac{3Q_1^{(2)}}{Q_1^{(1)2}} - \frac{6Q_1^{(2)2}}{Q_1^{(1)4}} + \frac{3Q_1^{(3)}}{Q_1^{(1)3}} \\
c_{2,1}^{(1)} &= \frac{Q_1^{(2)}}{Q_1^{(1)2}} - \frac{4Q_1^{(2)2}}{Q_1^{(1)4}} + \frac{3Q_1^{(3)}}{Q_1^{(1)3}} \\
c_{3,0}^{(1)} &= \frac{10Q_1^{(2)}}{3Q_1^{(1)2}} - \frac{18Q_1^{(2)2}}{Q_1^{(1)4}} + \frac{80Q_1^{(2)3}}{3Q_1^{(1)6}} + \frac{8Q_1^{(3)}}{Q_1^{(1)3}} - \frac{24Q_1^{(2)}Q_1^{(3)}}{Q_1^{(1)5}} \\
c_{3,1}^{(1)} &= \frac{3Q_1^{(2)}}{Q_1^{(1)2}} - \frac{22Q_1^{(2)2}}{Q_1^{(1)4}} + \frac{40Q_1^{(2)3}}{Q_1^{(1)6}} + \frac{13Q_1^{(3)}}{Q_1^{(1)3}} - \frac{42Q_1^{(2)}Q_1^{(3)}}{Q_1^{(1)5}} \\
c_{3,2}^{(1)} &= \frac{Q_1^{(2)}}{Q_1^{(1)2}} - \frac{8Q_1^{(2)2}}{Q_1^{(1)4}} + \frac{16Q_1^{(2)3}}{Q_1^{(1)6}} + \frac{5Q_1^{(3)}}{Q_1^{(1)3}} - \frac{18Q_1^{(2)}Q_1^{(3)}}{Q_1^{(1)5}} \\
c_{1,0}^{(2)} &= \frac{3Q_1^{(3)}}{Q_1^{(1)}Q_1^{(2)}} \\
c_{2,0}^{(2)} &= \frac{-27Q_1^{(3)}}{2Q_1^{(1)3}} + \frac{9Q_1^{(3)}}{Q_1^{(1)}Q_1^{(2)}} - \frac{9Q_1^{(3)2}}{2Q_1^{(1)2}Q_1^{(2)2}} \\
c_{2,1}^{(2)} &= \frac{-6Q_1^{(3)}}{Q_1^{(1)3}} + \frac{2Q_1^{(3)}}{Q_1^{(1)}Q_1^{(2)}}
\end{aligned}
\tag{5.4}
$$

$$
\begin{aligned}
c^{(2)}_{3,0} &= \frac{-1377 Q^{(3)}_1}{16 Q^{(1)}_1{}^3} + \frac{171 Q^{(3)}_1}{8 Q^{(1)}_1 Q^{(2)}_1} + \frac{243 Q^{(2)}_1 Q^{(3)}_1}{2 Q^{(1)}_1{}^5} - \frac{27 Q^{(3)}_1{}^2}{Q^{(1)}_1{}^2 Q^{(2)}_1{}^2} \\
&\quad - \frac{81 Q^{(3)}_1{}^2}{16 Q^{(1)}_1{}^4 Q^{(2)}_1} + \frac{9 Q^{(3)}_1{}^3}{Q^{(1)}_1{}^3 Q^{(2)}_1{}^3} \\
c^{(2)}_{3,1} &= \frac{-135 Q^{(3)}_1}{2 Q^{(1)}_1{}^3} + \frac{12 Q^{(3)}_1}{Q^{(1)}_1 Q^{(2)}_1} + \frac{108 Q^{(2)}_1 Q^{(3)}_1}{Q^{(1)}_1{}^5} - \frac{6 Q^{(3)}_1{}^2}{Q^{(1)}_1{}^2 Q^{(2)}_1{}^2} - \frac{45 Q^{(3)}_1{}^2}{2 Q^{(1)}_1{}^4 Q^{(2)}_1} \\
c^{(2)}_{3,2} &= \frac{-13 Q^{(3)}_1}{Q^{(1)}_1{}^3} + \frac{2 Q^{(3)}_1}{Q^{(1)}_1 Q^{(2)}_1} + \frac{24 Q^{(2)}_1 Q^{(3)}_1}{Q^{(1)}_1{}^5} - \frac{9 Q^{(3)}_1{}^2}{Q^{(1)}_1{}^4 Q^{(2)}_1}.
\end{aligned} \tag{5.5}
$$

$sl(4)$ case

$$
\begin{aligned}
c^{(1)}_{1,0} &= \frac{2 Q^{(2)}_1}{Q^{(1)}_1{}^2} \\
c^{(1)}_{2,0} &= \frac{3 Q^{(2)}_1}{Q^{(1)}_1{}^2} - \frac{6 Q^{(2)}_1{}^2}{Q^{(1)}_1{}^4} + \frac{3 Q^{(3)}_1}{Q^{(1)}_1{}^3} \\
c^{(1)}_{2,1} &= \frac{Q^{(2)}_1}{Q^{(1)}_1{}^2} - \frac{4 Q^{(2)}_1{}^2}{Q^{(1)}_1{}^4} + \frac{3 Q^{(3)}_1}{Q^{(1)}_1{}^3} \\
c^{(1)}_{3,0} &= \frac{10 Q^{(2)}_1}{3 Q^{(1)}_1{}^2} - \frac{18 Q^{(2)}_1{}^2}{Q^{(1)}_1{}^4} + \frac{80 Q^{(2)}_1{}^3}{3 Q^{(1)}_1{}^6} + \frac{8 Q^{(3)}_1}{Q^{(1)}_1{}^3} - \frac{24 Q^{(2)}_1 Q^{(3)}_1}{Q^{(1)}_1{}^5} + \frac{4 Q^{(4)}_1}{Q^{(1)}_1{}^4} \\
c^{(1)}_{3,1} &= \frac{3 Q^{(2)}_1}{Q^{(1)}_1{}^2} - \frac{22 Q^{(2)}_1{}^2}{Q^{(1)}_1{}^4} + \frac{40 Q^{(2)}_1{}^3}{Q^{(1)}_1{}^6} + \frac{13 Q^{(3)}_1}{Q^{(1)}_1{}^3} - \frac{42 Q^{(2)}_1 Q^{(3)}_1}{Q^{(1)}_1{}^5} + \frac{8 Q^{(4)}_1}{Q^{(1)}_1{}^4} \\
c^{(1)}_{3,2} &= \frac{Q^{(2)}_1}{Q^{(1)}_1{}^2} - \frac{8 Q^{(2)}_1{}^2}{Q^{(1)}_1{}^4} + \frac{16 Q^{(2)}_1{}^3}{Q^{(1)}_1{}^6} + \frac{5 Q^{(3)}_1}{Q^{(1)}_1{}^3} - \frac{18 Q^{(2)}_1 Q^{(3)}_1}{Q^{(1)}_1{}^5} + \frac{4 Q^{(4)}_1}{Q^{(1)}_1{}^4}
\end{aligned} \tag{5.6}
$$

$$
\begin{aligned}
c^{(2)}_{1,0} &= \frac{3 Q^{(3)}_1}{Q^{(1)}_1 Q^{(2)}_1} \\
c^{(2)}_{2,0} &= \frac{-27 Q^{(3)}_1}{2 Q^{(1)}_1{}^3} + \frac{9 Q^{(3)}_1}{Q^{(1)}_1 Q^{(2)}_1} - \frac{9 Q^{(3)}_1{}^2}{2 Q^{(1)}_1{}^2 Q^{(2)}_1{}^2} + \frac{9 Q^{(4)}_1}{Q^{(1)}_1{}^2 Q^{(2)}_1} \\
c^{(2)}_{2,1} &= \frac{-6 Q^{(3)}_1}{Q^{(1)}_1{}^3} + \frac{2 Q^{(3)}_1}{Q^{(1)}_1 Q^{(2)}_1} + \frac{4 Q^{(4)}_1}{Q^{(1)}_1{}^2 Q^{(2)}_1} \\
c^{(2)}_{3,0} &= \frac{-1377 Q^{(3)}_1}{16 Q^{(1)}_1{}^3} + \frac{171 Q^{(3)}_1}{8 Q^{(1)}_1 Q^{(2)}_1} + \frac{243 Q^{(2)}_1 Q^{(3)}_1}{2 Q^{(1)}_1{}^5} - \frac{27 Q^{(3)}_1{}^2}{Q^{(1)}_1{}^2 Q^{(2)}_1{}^2} \\
&\quad - \frac{81 Q^{(3)}_1{}^2}{16 Q^{(1)}_1{}^4 Q^{(2)}_1} + \frac{9 Q^{(3)}_1{}^3}{Q^{(1)}_1{}^3 Q^{(2)}_1{}^3} - \frac{81 Q^{(4)}_1}{Q^{(1)}_1{}^4} + \frac{783 Q^{(4)}_1}{16 Q^{(1)}_1{}^2 Q^{(2)}_1} - \frac{27 Q^{(3)}_1 Q^{(4)}_1}{Q^{(1)}_1{}^3 Q^{(2)}_1{}^2}
\end{aligned}
$$

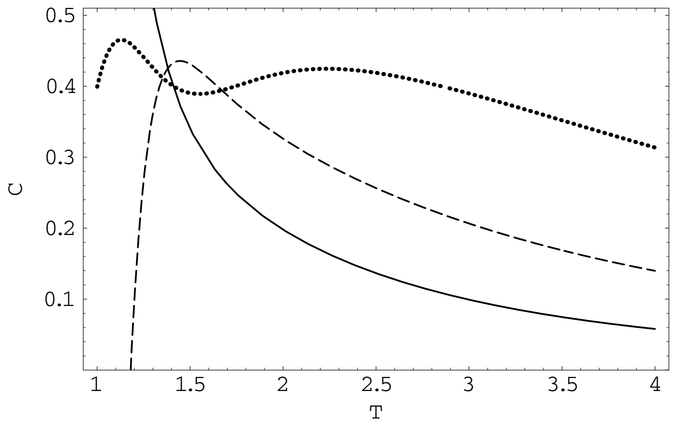

Figure 4. Temperature dependence of the high-temperature expansion of the specific heat of order 8 for the $sl(3)$, $J = 1$, $\mu_1 = h$, $\mu_2 = 0$, $\mu_3 = -h$ case ($h = 0$, smooth line; $h = 2$, broken line; $h = 4$, dotted line).

$$\begin{aligned}
c^{(2)}_{3,1} &= \frac{-135 Q_1^{(3)}}{2{Q_1^{(1)}}^3} + \frac{12 Q_1^{(3)}}{Q_1^{(1)} Q_1^{(2)}} + \frac{108 Q_1^{(2)} Q_1^{(3)}}{{Q_1^{(1)}}^5} - \frac{6{Q_1^{(3)}}^2}{{Q_1^{(1)}}^2 {Q_1^{(2)}}^2} \\
&\quad - \frac{45{Q_1^{(3)}}^2}{2{Q_1^{(1)}}^4 Q_1^{(2)}} - \frac{72 Q_1^{(4)}}{{Q_1^{(1)}}^4} + \frac{75 Q_1^{(4)}}{2{Q_1^{(1)}}^2 Q_1^{(2)}} - \frac{12 Q_1^{(3)} Q_1^{(4)}}{{Q_1^{(1)}}^3 {Q_1^{(2)}}^2} \\
c^{(2)}_{3,2} &= \frac{-13 Q_1^{(3)}}{{Q_1^{(1)}}^3} + \frac{2 Q_1^{(3)}}{Q_1^{(1)} Q_1^{(2)}} + \frac{24 Q_1^{(2)} Q_1^{(3)}}{{Q_1^{(1)}}^5} - \frac{9{Q_1^{(3)}}^2}{{Q_1^{(1)}}^4 Q_1^{(2)}} \\
&\quad - \frac{16 Q_1^{(4)}}{{Q_1^{(1)}}^4} + \frac{7 Q_1^{(4)}}{{Q_1^{(1)}}^2 Q_1^{(2)}}
\end{aligned} \tag{5.7}$$

$$\begin{aligned}
c^{(3)}_{1,0} &= \frac{4 Q_1^{(4)}}{Q_1^{(1)} Q_1^{(3)}} \\
c^{(3)}_{2,0} &= \frac{20 Q_1^{(4)}}{Q_1^{(1)} Q_1^{(3)}} - \frac{32 Q_1^{(2)} Q_1^{(4)}}{{Q_1^{(1)}}^3 Q_1^{(3)}} - \frac{8{Q_1^{(4)}}^2}{{Q_1^{(1)}}^2 {Q_1^{(3)}}^2} \\
c^{(3)}_{2,1} &= \frac{3 Q_1^{(4)}}{Q_1^{(1)} Q_1^{(3)}} - \frac{8 Q_1^{(2)} Q_1^{(4)}}{{Q_1^{(1)}}^3 Q_1^{(3)}} \\
c^{(3)}_{3,0} &= \frac{-192 Q_1^{(4)}}{{Q_1^{(1)}}^4} + \frac{248 Q_1^{(4)}}{3 Q_1^{(1)} Q_1^{(3)}} - \frac{352 Q_1^{(2)} Q_1^{(4)}}{{Q_1^{(1)}}^3 Q_1^{(3)}} + \frac{512 {Q_1^{(2)}}^2 Q_1^{(4)}}{{Q_1^{(1)}}^5 Q_1^{(3)}} \\
&\quad - \frac{80{Q_1^{(4)}}^2}{{Q_1^{(1)}}^2 {Q_1^{(3)}}^2} + \frac{128 Q_1^{(2)} {Q_1^{(4)}}^2}{{Q_1^{(1)}}^4 {Q_1^{(3)}}^2} + \frac{64{Q_1^{(4)}}^3}{3{Q_1^{(1)}}^3 {Q_1^{(3)}}^3} \\
c^{(3)}_{3,1} &= \frac{-96 Q_1^{(4)}}{{Q_1^{(1)}}^4} + \frac{30 Q_1^{(4)}}{Q_1^{(1)} Q_1^{(3)}} - \frac{160 Q_1^{(2)} Q_1^{(4)}}{{Q_1^{(1)}}^3 Q_1^{(3)}} + \frac{256 {Q_1^{(2)}}^2 Q_1^{(4)}}{{Q_1^{(1)}}^5 Q_1^{(3)}}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{12 Q_1^{(4)^2}}{Q_1^{(1)^2} Q_1^{(3)^2}} + \frac{32 Q_1^{(2)} Q_1^{(4)^2}}{Q_1^{(1)^4} Q_1^{(3)^2}} \\
c_{3,2}^{(3)} &= \frac{-12 Q_1^{(4)}}{Q_1^{(1)^4}} + \frac{3 Q_1^{(4)}}{Q_1^{(1)} Q_1^{(3)}} - \frac{18 Q_1^{(2)} Q_1^{(4)}}{Q_1^{(1)^3} Q_1^{(3)}} + \frac{32 Q_1^{(2)^2} Q_1^{(4)}}{Q_1^{(1)^5} Q_1^{(3)}}.
\end{aligned}
\tag{5.8}
$$

When $Q_1^{(a)}$ are suitably chosen, equations (5.3) recover the known results [30]. We see that these coefficients are expressed in terms of the solutions of the Q-system (3.4). Thus, a solution of the T-system is given in terms of solutions of the Q-system. The T-system is a Yang–Baxterization of the Q-system. Thus, we may say that the degree of the expansion expresses the degree of a Yang–Baxterization. Using $\left\{b_n^{(a)}(0)\right\}$, we can calculate the free energy (4.8) and the specific heat $C = -T\frac{\partial^2 f}{\partial T^2}$. We have plotted the high-temperature expansion of the specific heat for $sl(3)$ in figure 4. For large T, this agrees with the result from another NLIE by Fujii and Klümper (see figure 4 in [18]). This indicates the validity of our new NLIE (4.7). We note that the high-temperature expansion for the $SU(n)$ Heisenberg model is briefly reported in [31] based on a completely different method.

6. Concluding remarks

In this paper, we have derived a system of NLIEs with a *finite* number of unknown functions, which describes the thermodynamics of the $sl(r+1)$ Uimin–Sutherland model. This type of NLIE for $sl(r+1)$ of *arbitrary* rank r is derived for the first time. In particular for $r = 1$, our new NLIE (4.7) reduces to Takahashi's NLIE [19] for the XXX spin chain. We clarify a relation between our new NLIEs (4.6) and (4.7) and the traditional TBA equations (3.11), (3.13) and (3.16), which are also derived from the T-system. The high-temperature expansion of the free energy is discussed, in which a solution of the T-system is given in terms of solutions of the Q-system. We expect that we can extend these results to other algebras by using the T-systems in [21, 32, 34–37, 23].

In [38], fugacity expansion formulae of the free energy for the XXX spin chain up to an infinite order are derived from the string centre equation, and Takahashi's NLIE [19] is rederived from these formulae. This work may be viewed in part as a type of Yang–Baxterization of the results in [39] where power series formulae related to a formal completeness of the Bethe ansatz are derived from the string centre equation. We can also generally recover [40] the results in [39] from the Q-system. Starting from our new NLIE (4.7), we may derive expansion formulae similar to those in [38] for $sl(r+1)$ of arbitrary rank r, from which an idea toward a Yang–Baxterization of the results in [40] may be given, since our new NLIE is based on the T-system which is a Yang–Baxterization of the Q-system.

There are different types of NLIEs with finite numbers of unknown functions for algebras of arbitrary rank in rather different contexts [41, 42]. Their origins are related to Destri and de Vega's NLIE [43]. The derivation of analogous NLIEs from the QTM method and the study of the relation to our new NLIE deserve investigation.

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